Cutoff Wavenumbers of Goubau Lines

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Abstract—**The cutoff wavenumbers k_{nm} of the surface wavemodes in a Goubau line are determined by a procedure that leads directly to the eigenvalue equation in the limit of zero values for the wavenumber of the infinite exterior medium. The same result has been obtained by a much longer procedure based on an extension of previous methods used in connection with simple dielectric waveguides. Numerical results and related curves for all types of modes are also included.**

I. Solution of the Eigenvalue Equation

THE WAVEGUIDE is shown in Fig. 1. The radius of the conductor is R_1, and the radius of the dielectric cladding with dielectric constant ϵ_1 is R_2. The modes, in general, are neither TM nor TE, but hybrid, having both E_z and H_z (longitudinal) components. They are called EH and/or HE modes [1]–[6].

Denoting the coordinate of a general point P by r,θ,z, we have the following expressions for $E_z(P)=E_z(r,\theta)$:

$$E_z(P)=\{[A_nJ_n(kr)+B_nN_n(kr)]\cos(n\theta)+[C_nJ_n(kr)+D_nN_n(kr)]\sin(n\theta)\}\cdot\exp i(\omega t-\beta z),\qquad n\geqslant 0, R_1\leqslant r\leqslant R_2 \tag{1}$$

$$E_z(P)=[A'_nH_n(i\gamma r)\cos(n\theta)+B'_nH_n(i\gamma r)\sin(n\theta)]\cdot\exp i(\omega t-\beta z),\qquad n\geqslant 0, r\geqslant R_2 \tag{2}$$

where J_n, N_n are the usual cylindrical Bessel functions, H_n is the cylindrical Hankel function of the first kind, β is the propagation constant, and k is the cutoff wavenumber. The relations between β,γ,k are

$$\gamma^2=\beta^2-k_2^2 \qquad k_2^2=\omega^2\epsilon_2\mu_2 \tag{3}$$

$$k^2=(\epsilon-1)k_2^2-\gamma^2 \qquad \epsilon=\epsilon_1/\epsilon_2. \tag{4}$$

Expressions identical to (1) and (2) can be written for $H_z(P)$ by using small letters a_n, b_n, b'_n, etc., for the coefficients.

Satisfaction of the boundary conditions at $r=R_1$ and $r=R_2$, which require continuity of tangential E- and H-field components, yields a system of twelve homogeneous equations for the coefficients. For nontrivial solutions, the determinant of this system must vanish. This provides the eigenvalue equation from which the cutoff wavenumbers k_{nm} can be determined.

This eigenvalue equation has the form [4]:

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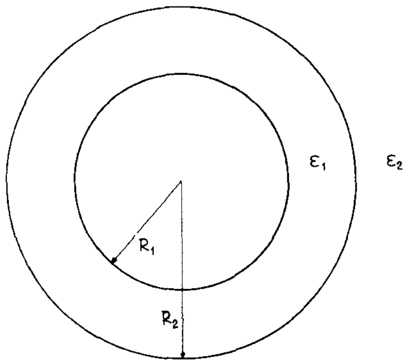

Fig. 1. The cross section of the guide.

$$\left[\frac{\epsilon R_2\gamma^2}{k}\frac{M'_n}{M_n}+i\gamma R_2\frac{H'_n(i\gamma R_2)}{H_n(i\gamma R_2)}\right]\cdot\left[\frac{R_2\gamma^2}{k}\frac{L'_n}{L_n}+i\gamma R_2\frac{H'_n(i\gamma R_2)}{H_n(i\gamma R_2)}\right]=\left[n(\epsilon-1)\frac{\beta k_2}{k^2}\right]^2 \tag{5}$$

where

$$L'_n=J'_n(x_2)N'_n(x_1)-J'_n(x_1)N'_n(x_2)$$
$$L_n=J_n(x_2)N'_n(x_1)-J'_n(x_1)N_n(x_2) \tag{6}$$
$$M'_n=J'_n(x_2)N_n(x_1)-J_n(x_1)N'_n(x_2)$$
$$M_n=J_n(x_2)N_n(x_1)-J_n(x_1)N_n(x_2) \tag{7}$$
$$x_1=kR_1$$
$$x_2=kR_2. \tag{8}$$

We shall find the solutions of (5) in the limit $\gamma\to 0$, corresponding to the cutoff condition. As pointed out in [4], a more practical cutoff condition for sufficient field confinement should be determined rather by $\gamma R_2\geqslant 0.001$. Further relative discussions and results can also be found in this basic reference. Here, however, we proceed by applying a Maclaurin expansion to (5) for small γ [5], [6]. In this way, we quickly obtain the general solution of (5) at cutoff without any restrictions on the values of the parameters. Thus for small γ:

$$i\gamma R_2\frac{H'_1(i\gamma R_2)}{H_1(i\gamma R_2)}=-1+\frac{1}{2}\gamma^2R_2^2\ln\frac{\Gamma\gamma R_2}{2} \qquad \Gamma=1.781072 \tag{9}$$

$$i\gamma R_2\frac{H'_n(i\gamma R_2)}{H_n(i\gamma R_2)}=-n-\frac{\gamma^2R_2^2}{2(n-1)}+0(\gamma^4),\qquad n>1. \tag{10}$$

For small γ and with the use of (3), (4) the right side of (5) becomes

$$n^2(\epsilon-1)^2\frac{\beta^2k_2^2}{k^4}=n^2(\epsilon-1)^2\frac{(\gamma^2+k_2^2)k_2^2}{[(\epsilon-1)k_2^2-\gamma^2]^2}$$

$$=n^2+\frac{(\epsilon+1)n^2}{(\epsilon-1)k_2^2}\gamma^2+0(\gamma^4). \quad (11)$$

Substituting in (5) and using (10) and (11) we find for small γ and $n>1$:

$$\frac{\epsilon R_2^2\gamma^4}{k^2}\frac{M_n'L_n'}{M_nL_n}+\frac{\epsilon R_2\gamma^2}{k}\frac{M_n'}{M_n}\left[-\frac{\gamma^2R_2^2}{2(n-1)}-n\right]$$

$$+\left[-\frac{\gamma^2R_2^2}{2(n-1)}-n\right]\frac{R_2\gamma^2}{k}\frac{L_n'}{L_n}+\frac{\gamma^4R_2^4}{4(n-1)^2}$$

$$+n^2+\frac{\gamma^2R_2^2n}{n-1}=n^2+\frac{(\epsilon+1)n^2}{(\epsilon-1)k_2^2}\gamma^2+0(\gamma^4). \quad (12)$$

After canceling n^2 from both sides, and multiplying (12) by $(k^2M_nL_n)/(\gamma^2)$, we find in the limit $\gamma\to 0$:

$$\epsilon x_2M_n'L_n(-n)+(-n)x_2L_n'M_n$$

$$+\frac{x_2^2n}{n-1}M_nL_n-(\epsilon+1)n^2L_nM_n=0$$

or

$$\epsilon x_2M_n'L_n+x_2L_n'M_n$$

$$-\left[\frac{x_2^2}{n-1}-(\epsilon+1)n\right]L_nM_n=0, \qquad n>1 \quad (13)$$

If $n=1$, an equation similar to (12) is obtained, but in place of (10) we must use (9). Eliminating again the constant term (1 in this case), and multiplying both sides of the equation by

$$\frac{k^2M_1L_1}{\gamma^2\ln\frac{\Gamma\gamma R_2}{2}}$$

we find in the limit $\gamma\to 0$:

$$-\frac{1}{\ln\frac{\Gamma\gamma R_2}{2}}\left[\epsilon x_2M_1'L_1+x_2L_1'M_1+M_1L_1(\epsilon+1)\right]$$

$$-x_2^2L_1M_1=0. \quad (14)$$

As $\gamma\to 0$ the term $\ln(\Gamma\gamma R_2/2)\to-\infty$, so in order to satisfy (14) the product $x_2^2L_1M_1$ must vanish:

$$x_2=0 \qquad L_1=0 \qquad M_1=0$$

or from (6) and (7)

$k=0(\mathrm{HE}_{11})$

$$J_1(x_2)N_1'(x_1)-J_1'(x_1)N_1(x_2)=0, \qquad \mathrm{HE}_{1m}, m>1$$

$$J_1(x_2)N_1(x_1)-J_1(x_1)N_1(x_2)=0, \qquad \mathrm{EH}_{1m}, m\geqslant 1. \quad (15)$$

When $n=0$ the right-hand side of (5) vanishes. In this case, we obtain two separate and independent eigenvalue equations corresponding to pure TM and TE modes:

$$\epsilon\frac{R_2\gamma^2}{k}\frac{M_0'}{M_0}+i\gamma R_2\frac{H_0'(i\gamma R_2)}{H_0(i\gamma R_2)}=0, \qquad \mathrm{TM}_{0m} \text{ or } E_{0m} \quad (16)$$

$$\frac{R_2\gamma^2}{k}\frac{L_0'}{L_0}+i\gamma R_2\frac{H_0'(i\gamma R_2)}{H_0(i\gamma R_2)}=0, \qquad \mathrm{TE}_{0m} \text{ or } H_{0m}. \quad (17)$$

For small values of γ

$$i\gamma R_2\frac{H_0'(i\gamma R_2)}{H_0(i\gamma R_2)}=\frac{1}{\ln(\Gamma\gamma R_2/2)} \quad (18)$$

and (16) becomes

$$\frac{kM_0}{\epsilon M_0'}=\gamma^2R_2\ln\frac{2}{\Gamma\gamma R_2}\underset{\gamma\to 0}{\to}0. \quad (19)$$

Therefore,

$$M_0=0 \quad \text{or} \quad J_0(x_2)N_0(x_1)-J_0(x_1)N_0(x_2)=0,$$

$$\mathrm{TM}_{0m} \quad \text{or} \quad E_{0m}. \quad (20)$$

Similarly the solution of (17) in the limit $\gamma=0$ is

$$L_0=0 \quad \text{or} \quad J_0(x_2)N_0'(x_1)-J_0'(x_1)N_0(x_2)=0,$$

$$\mathrm{TE}_{0m} \quad \text{or} \quad H_{0m}. \quad (21)$$

Exactly the same results were found also following an extension of the laborious method used in [1]–[3] for simple dielectric waveguides. In particular, the abbreviations

$$J^+=\frac{1}{x_2^2}\left[n-x_2\frac{M_n'}{M_n}\right] \qquad J^-=\frac{1}{x_2^2}\left[n+x_2\frac{M_n'}{M_n}\right]$$

$$N^+=\frac{1}{x_2^2}\left[n-x_2\frac{L_n'}{L_n}\right] \qquad N^-=\frac{1}{x_2^2}\left[n+x_2\frac{L_n'}{L_n}\right]$$

$$H^+=\frac{1}{i\gamma R_2}\frac{H_{n+1}(i\gamma R_2)}{H_n(i\gamma R_2)} \qquad H^-=\frac{1}{i\gamma R_2}\frac{H_{n-1}(i\gamma R_2)}{H_n(i\gamma R_2)} \quad (22)$$

are the proper ones to use to reduce (5) to (13) or (15) for $n\geqslant 2$ or $n=1$, following the very long procedure based on such symbols, and developed in [1], [2]. We notice also, that the simple procedure of expanding both sides of the original eigenvalue equation in as many powers of γ as necessary (the first two suffice) developed here is applicable to the simpler problems of [1]–[3], and in a straightforward manner yields the same results as theirs.

It should also be mentioned that in the limit $R_1\to 0$ all our results reduce analytically to the corresponding ones for the simple dielectric rod [1], [3].

II. Numerical Results

In Tables I–III we give the cutoff values $x_2=kR_2$ of certain Goubau lines. For $n=0, 1$, the results are independent of ϵ. The cutoff frequency of the HE_{11} mode is again 0, as seen from (15). The designation HE_{1m} or EH_{1m} for the $n=1$ modes in (15) is adopted as an evident extension of the corresponding designations in (20) and (21), i.e., HE corresponds to TE or H modes and EH to TM or E

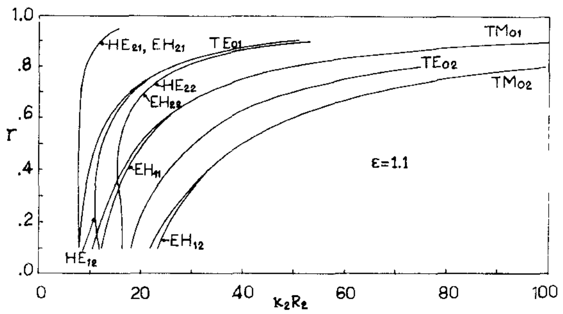

Fig. 2. Modal loci of r versus k_2R_2 for $\epsilon=1.1$.

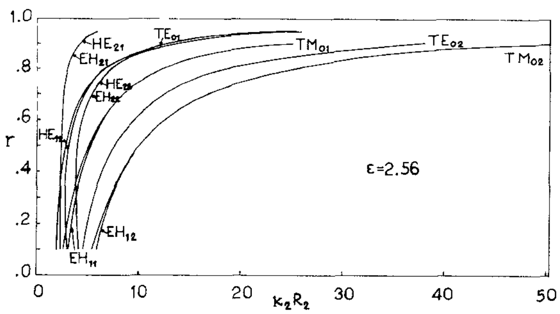

Fig. 3. Modal loci of r versus k_2R_2 for $\epsilon=2.56$.

TABLE I
VALUES OF x_2 FOR $r=R_1/R_2=0.7$

	m=1	m=2	m=3	m=4		
n=0	5.6188301	15.846498	26.263696	36.711868		TE
0	10.455235	20.935466	31.410255	41.883644		TM
1	0.0000000	5.7614168	15.892475	26.291096		HE
1	10.522032	20.969386	31.432936	41.900673		EH
2	3.3235439	6.5833374	11.447004	16.229897	ε=2.56	EH
3	4.7582362	7.4430970	12.332795	16.640061	ε=2.56	HE
2	3.7861071	6.5786861	11.804340	16.229639	ε=4	EH
3	5.1941428	7.4294928	12.873836	16.639063	ε=4	HE

TABLE II
VALUES OF x_2 FOR $r=R_1/R_2=0.8$

	m=1	m=2	m=3	m=4		
n=0	8.2121663	23.687132	39.345330	55.031805		TE
0	15.698088	31.410962	47.120577	62.829367		TM
1	0.0000000	8.2945213	23.713813	39.361280		HE
1	15.737552	31.430815	47.133827	62.839309		EH
2	3.8449490	9.0395070	16.617158	23.999615	ε=2.56	EH
3	5.4027881	9.8092407	17.467882	24.330011	ε=2.56	HE
2	4.4931387	9.0380297	17.009093	23.999547	ε=4	EH
3	6.0740286	9.8043928	18.123385	24.329746	ε=4	HE

TABLE III
VALUES OF x_2 FOR $r=R_1/R_2=0.9$

	m=1	m=2	m=3	m=4		
n=0	16.044864	47.238528	78.608717	110.00498		TE
0	31.411513	62.829643	94.246306	125.66260		TM
1	0.0000000	16.080907	47.250348	78.615798		HE
1	31.429164	62.838481	94.252200	125.66702		EH
2	5.2068404	16.766003	32.273123	47.495533	ε=2.56	EH
3	7.2713189	17.463501	33.103361	47.761411	ε=2.56	HE
2	6.2779388	16.765814	32.699290	47.495525	ε=4	EH
3	8.5490545	17.462803	33.888957	47.761381	ε=4	HE

modes. For $n \geq 2$ the cutoff wavenumber is unique and can be designated as either EH or HE.

In Figs. 2 and 3 modal loci are drawn providing the variation of $r = R_1/R_2$ versus k_2R_2 for various values of ϵ. As $r \to 1$ the cutoff values k_2R_2 tend to ∞, as expected for the case of a perfectly conducting rod.

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A Study of Waveguides for Far Infrared Interferometers Measuring Electron Density of Tokamak Plasmas

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***Abstract*—In the 0.1–1-mm wavelength range, waveguide propagation offers some advantages over optical propagation in multichannel infrared interferometers measuring electron density of Tokamak plasmas. In this paper, the necessary conditions for use of waveguides for this purpose are defined. Possible waveguides are theoretically and experimentally studied, taking into account their shape, size, material, and length. It is shown that it is possible to find waveguides well suited for these interferometers. These results can also be applied to other far infrared interferometers and devices.**

I. Introduction

BECAUSE OF refraction effects measurement of electron density with multichannel interferometers in large Tokamaks requires the use of infrared radiation rather than microwaves [1]–[3]. For example, an eight-channel interferometer is in operation on the TFR Tokamak, using a wavelength of $\lambda = 0.337$ mm [4]. Up to now infrared interferometers have used free-space propagation of beams. However, in large Tokamaks, beam paths are several meters long, and, therefore, require the use of large optical components due to beam deviation and divergence effects which are always present. A waveguide device would be less sensitive to these undesirable effects, and moreover, would be easier to realize [5]. However, waveguide propagation must satisfy two essential conditions:

1) the propagation must avoid excessive losses,
2) the wave polarization (i.e., the direction of the electric field lines) must be linear [4].

These conditions must be satisfied even for small deviations of the beam direction at the input of the waveguide. Here we will study several waveguide structures in the 0.1–1-mm wavelength range, since this is the interesting range for interferometers used on large Tokamaks.

II. Choice of Waveguide Structure

Among the different possible waveguides, optic fibers and open or closed H guides [6]–[8] are less attractive than oversized closed waveguides (i.e., hollow waveguides with dielectric or metallic walls), for two reasons.

1) Open waveguides radiate some energy and this may result in stray signals in neighboring waveguides.

2) Attenuation in these waveguides is not as small as that of greatly oversized waveguides, because a part of the wave propagates inside a dielectric which always has some loss.

Among oversized closed waveguides, it is possible to make a distinction between those having dielectric walls and metallic walls. For most metals at microwave and far infrared wavelengths it is possible to neglect the real component of the refractive index and, therefore, consider only the imaginary component when calculating the attenuation. In the case of a dielectric, such as ordinary glass [10], it can be shown that for the lowest order mode

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